Design Sensitivity Analysis of Beams Under Nonlinear Forced Vibrations

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Abstract

A COMPUTATIONAL method for design sensitivity analysis of an eigenvalue and an eigenvector of a beam under nonlinear forced vibration is presented in this paper. The nonlinear vibration problem is only analyzed once in the proposed method. The geometric nonlinearity of concern results from the large deflection of a beam. The finite element system equation for nonlinear vibration is symmetric. However, it is found that the equation for computing the design sensitivity of an eigenvector is linear and unsymmetric. A numerical example is included to validate the proposed computational procedure.

Contents

The nonlinear vibrations studied herein refer to the periodic (though not necessarily harmonic) and stable motions of a nonlinear system.

Recently, Hou and Yuan[1] addressed the design sensitivity analysis of eigenvalues and eigenvectors of a beam under nonlinear free vibration. In that paper, the longitudinal inertia was not considered in the nonlinear vibration formulation, so that the problem could be expressed in terms of the lateral deflection alone. In the present work the design sensitivity analysis has been extended successfully to a nonlinear forced vibration problem whose formulation includes the in-plane inertia and displacement. The variational formulation[2] for the nonlinear forced vibrations of beams under harmonic excitation is given as

$$\int_0^\ell EIw,_{xx}z,_{xx}\,\mathrm{d}x + \int_0^\ell [EA\,(u,_x + fw,_x)(v,_x + fz,_x)]\,\mathrm{d}x - \lambda\int_0^\ell m\,(wz + uv)\,\mathrm{d}x - B\int_0^\ell wz\,\mathrm{d}x = 0 \tag{1}$$

where $u(x)$ denotes the longitudinal deformation, $f = (1/2)\,w,_x$, and $z(x)$ and $v(x)$ are test functions for $w(x)$ and $u(x)$, respectively. The last term in Eq. (1) is the potential energy of a linear spring force that is an approximation of the virtual work done by the harmonic excitation. Note that the variational formulation of a beam nonlinear free vibration is the same as Eq. (1) with $B = 0$.

The spring constant B in Eq. (1) can be derived to be

$$B = \frac{cF_0}{\gamma} \tag{2a}$$

$$c = \int_{\ell_1}^{\ell_2} \phi\,\mathrm{d}x \Big/ \int_0^\ell \phi^2\,\mathrm{d}x \tag{2b}$$

where F_0 is the actual applied force intensity, γ is the maximum amplitude of the normalized eigenmode ϕ, and the harmonic force $F_0 \cos\sqrt{\lambda}t$ is applied on the beam from $x = \ell_1$ to ℓ_2. The finite-element equation corresponding to Eq. (1) can be obtained as

$$[K]\{x\} + [G(\{x\})]\{x\} - [H(\{x\})]\{x\} - \lambda[M]\{x\} = 0 \tag{3}$$

The symmetric matrices $[K]$, $[G]$, and $[M]$ are the linear stiffness matrix, the nonlinear geometric matrix, and the mass matrix, respectively. The harmonic force matrix $[H]$ is evaluated according to the last term of Eq. (1). Moreover, this matrix is symmetric and nonlinear in terms of $\{x\}$ because of the coefficient c defined in Eq. (2). It is interesting to note that the nonlinear forced vibration problem can be formulated as a nonlinear free-vibration problem. It is a sharp contrast to the problem of linear forced-vibration.

For simplicity, the coefficient matrices of the finite-element equation of Eq. (1) can be combined to form a simple equation as

$$[A]\{x\} = 0 \tag{4}$$

where $[A] = [K] + [\bar{G}] - \lambda[M]$, and the nonlinear matrix $[\bar{G}] = [G] - [H]$. Premultiplying eigenvector $\{x\}^T$ to Eq. (4), one immediately has the following equality:

$$\{x\}^T\,[A]\{x\} = 0 \tag{5}$$

Let $[E]$ denote the derivative of the vector $([\bar{G}(\{x\})]\{x\})$ in which the postmultiplying eigenvector $\{x\}$ is held fixed with respect to the eigenvector. Furthermore, let the subscript b denote the design derivative. Then, it is straightforward to obtain the design derivative of Eq. (4) as

$$([A] + [E])\{x_b\} - \lambda_b[M]\{x\} = -[A_b]\{x\} \tag{6}$$

where $[A_b] = [K_b] + [\bar{G}_b] - \lambda[M_b]$. It is noted that $[E] = 0$ for a linear-vibration problem. Additionally, $[E]$ is a symmetric matrix for a nonlinear free-vibration problem, but is unsymmetric for a nonlinear forced-vibration problem. Equation (6) yields only n equations, but there are $n + 1$ unknowns for $\{x_b\}$ and λ_b. The additional relation can be generated by taking the design derivative of Eq. (5). It yields

$$\lambda_b = \frac{1}{\{x\}^T\,[M]\{x\}}\,[\{x\}^T\,[A_b]\{x\} + \{x\}^T\,[E]\{x_b\}] \tag{7}$$

It is in contrast to the linear, symmetric eigenvalue problem, as indicated in Eq. (7), that the design derivative of eigenvalue

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λ_b cannot be calculated without knowing the design derivative of eigenvector $\{x_b\}$ in advance. Substituting Eq. (7) into Eq. (6) for λ_b, one obtains a single equation of $\{x_b\}$ as

$$\left([A] + [E] - \frac{[M]\{x\}\{x\}^T [E]}{\{x\}^T [M]\{x\}}\right)\{x_b\} = -[A_b]\{x\} + \frac{[M]\{x\}}{\{x\}^T [M]\{x\}} \{x\}^T [A_b]\{x\} \tag{8}$$

or, symbolically, one can simplify Eq. (8) as

$$[B]\{x_b\} = \{c\} \tag{9}$$

One can easily show tht the eigenvector $\{x\}$ satisfies both $\{x\}^T [B]\{x_b\} = 0$, and $\{x\}^T \{c\} = 0$. Hence, the eigenvector $\{x\}$ is the orthogonal complement of the range of $[B]$ and at the same time in the null space of $[B]^T$. Based on the alternative theorem, it can be concluded that $[B]$ is singular. In order to avoid this singularity, the following linear combination is introduced

$$\{x_b\} = \{\bar{x}\} + \alpha\{x\} \tag{10}$$

where $\{\bar{x}\}$ is orthogonal to $\{x\}$ with respect to the mass matrix. In Eq. (10), $\{x\}$ may be considered as the homogenous solution and $\{\bar{x}\}$ as the particular solution. Finally, the design derivative of the nonlinear eigenvector can be obtained by solving the following matrix equations:

$$([A] + [E])\{\bar{x}\} = -[A_b]\{x\} - \alpha[E]\{x\} \tag{11}$$

$$\{\bar{x}\}^T [M]\{x\} = 0 \tag{12}$$

A simple way to solve a linear equation (11) with a linear constraint (12) is to apply the theorem of Lagrange multipliers to Eq. (11)

$$([A] + [E])\{\bar{x}\} + [M]\{x\}\mu = -[A_b]\{x\} - \alpha[E]\{x\} \tag{13}$$

where μ is a scalar representing the Lagrange multiplier. The combination of Eqs. (12) and (13) yields $n + 1$ simultaneous equations for $\{\bar{x}\}$ and μ

$$\begin{bmatrix} [A] + [E] & [M]\{x\} \\ \{x\}^T [M] & 0 \end{bmatrix} \begin{Bmatrix} \{\bar{x}\} \\ \mu \end{Bmatrix} = \begin{Bmatrix} -[A_b]\{x\} \\ 0 \end{Bmatrix} - \alpha \begin{Bmatrix} [E]\{x\} \\ 0 \end{Bmatrix} \tag{14}$$

It is noted that the leading coefficient matrix on the left side of Eq. (14) is nonsymmetric and positive-definite, and there is an undetermined coefficient α appearing on the right side. Since Eq. (14) is linear, one may superpose the solution as

$$\begin{Bmatrix} \{\bar{x}\} \\ \mu \end{Bmatrix} = \begin{Bmatrix} \{\bar{x}_1\} \\ \mu_1 \end{Bmatrix} + \alpha \begin{Bmatrix} \{\bar{x}_2\} \\ \mu_2 \end{Bmatrix}$$

where the first and the second terms are the solutions of Eq. (14) with $\{(-[A_b]\{x\})^T, 0\}^T$ and $\{(-[E]\{x\})^T, 0\}^T$ as the forcing terms, respectively.

The coefficient α remains to be determined to complete the computation of the eigenvector design derivative

$$\{x_b\} = \{\bar{x}\} + \alpha\{x\} = \{\bar{x}_1\} + \alpha(\{\bar{x}_2\} + \{x\}) \tag{15}$$

Table 1 Design sensitivity analysis of eigenvectors (displacements) associated with various perturbations of design variable for a clamped beam under harmonic loading

Percentage change ΔH, %	Actual change $\Delta w(\times 10^{-4}$ in.)	Prediction $w_h \times \Delta H(\times 10^{-4})$	Error, %[a]
−30.0	8.096	5.574	31.2
−18.0	4.158	3.344	19.6
−9.0	1.863	1.672	10.3
−3.0	0.579	0.557	3.8
+3.0	−0.542	−0.557	2.8
+9.0	−1.523	−1.672	9.8
+18.0	−2.780	−3.344	20.3
+30.0	−4.132	−5.574	34.9
CPU time[b]	13.75 s[c]	1.98 s[d]	

[a]Error $= |(w_h \times \Delta H - \Delta w)/\Delta w| \times 100\%$. [b]On CDC 830, single precision. [c]Analysis at $H = 0.064$ in. [d]Design sensitivity analysis at $H = 0.064$ in.

The coefficient α is usually determined by considering the normalization of the corresponding eigenvector. The eigenvectors of the nonlinear vibration problems are usually normalized with respect to the maximum amplitude, i.e., $\gamma = \max|\{x\}|$, where the constant γ is the assigned maximum amplitude. For simplicity, $\{x\}$ is assumed to have the maximum value at the fixed point x_0 in this study. Then it follows that $x(x_0) = \gamma$ and $dx(x_0)/db = 0$. This condition along with Eq. (15) yield the following simple relation in order to calculate α as

$$\alpha = -\frac{\bar{x}_1 (x_0)}{\bar{x}_2 (x_0) + x (x_0)} \tag{16}$$

In short, it can be concluded that with a given eigensolution $(\lambda, \{x\})$, the design derivative of eigenvector $\{x_b\}$ can be determined by solving Eqs. (14–16) for $\{\bar{x}_1\}$, $\{\bar{x}_2\}$, and α. Subsequently, the design derivative of eigenvalue λ_b can then be computed based on Eq. (7).

The example considered here is the design derivative of a clamped-clamped beam under forced vibration. The amplitude of the harmonic excitation is given as $F_0 = 0.3276$ lb/in. For the changes of beam thickness ranging from +30% to −30%, as shown in Table 1, the predicted changes of eigenvector and eigenvalue are in good agreement with the actual changes calculated by the finite difference. These results demonstrate the validity of the proposed computational procedure for the design sensitivity analysis of nonlinear vibrations.

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